Quantum phase transitions in alternating transverse Ising chains: Rigorous analytical and exact numerical results

Oleg Derzhko[1,2], Johannes Richter[3], Taras Krokhmalskii[1] and Oles' Zaburannyi[1]

[1]Institute for Condensed Matter Physics,

1 Svientsitskii Str., L'viv-11, 79011, Ukraine

[2]Chair of Theoretical Physics, Ivan Franko National University of L'viv,

12 Drahomanov Str., L'viv-5, 79005, Ukraine

[3]Institut für Theoretische Physik, Universität Magdeburg,

P.O. Box 4120, D-39016 Magdeburg, Germany

February 1, 2008

Abstract

We examine the ground state properties of the $s = \frac{1}{2}$ transverse Ising chain with regularly alternating bonds and fields using exact analytical results and exact numerical data for long (up to $N = 900$) and short ($N = 20$) chains. For a given period of alternation the system may exhibit a series of quantum phase transitions, where the number of transitions depends on the concrete set of the parameters of the Hamiltonian. The critical behaviour for the nonuniform and the uniform chain is the same.

PACS number(s): 75.10.-b

Keywords: Quantum Ising chain; Quantum phase transition

Currently the investigation of quantum phase transitions is a very active field of physical research [1, 2]. Among others the quantum magnets represent an interesting class of systems studied theoretically and experimentally where quantum phase transitions appear [2, 3, 4]. The one-dimensional $s = \frac{1}{2}$ Ising model in a transverse field (the transverse Ising chain) described by the Hamiltonian

$$H = \sum_n \Omega s_n^z + \sum_n 2I s_n^x s_{n+1}^x \tag{1}$$

($s_n^\alpha = \frac{1}{2}\sigma_n^\alpha$, σ_n^α are the Pauli matrices attached to the site n, Ω is the on-site transverse field, $2I$ is the exchange interaction between neighbouring spins) is known to be the simplest system exhibiting a quantum phase transition.

For positive Ω the system exhibits a zero temperature transition from the ordered (quantum Ising) phase (for $\Omega < \Omega_c$) to the disordered (quantum paramagnetic) phase (for $\Omega > \Omega_c$) at a critical point $\Omega_c = |I|$ [2]. (Notice, that the same kind of transition takes place at $\Omega_c = -|I|$.) There has been a great deal of theoretical work on the ground state (GS) properties of the transverse Ising chain (1). The knowledge of the eigenvalues and eigenfunctions

of the Hamiltonian (1) [5, 6, 2] makes the problem accessible for exact analysis. Let us briefly summarize some of the main exact results for the GS and the excitation gap. (i) At the critical transverse field Ω_c the gap in the energy spectrum Δ closes as $\Delta \sim |\Omega - \Omega_c|$. (ii) The GS correlation function $\langle s_j^x s_{j+n}^x \rangle$ tends to $m^{x2} \neq 0$ as $n \to \infty$ for $0 \leq \Omega < \Omega_c$ with the longitudinal (Ising) magnetization $m^x = \frac{1}{2} \left(1 - \left(\frac{\Omega}{\Omega_c} \right)^2 \right)^{\frac{1}{8}}$ which plays the role of the order parameter. (iii) For $\Omega > \Omega_c$ the correlation function $\langle s_j^x s_{j+n}^x \rangle$ decays exponentially to zero for large n. (iv) Close to the critical point Ω_c the transverse magnetization m^z and the static transverse susceptibility χ^z behave as $m^z \sim (\Omega - \Omega_c) \ln |\Omega - \Omega_c|$ and $\chi^z \sim \ln |\Omega - \Omega_c|$, respectively. (v) At the critical point $\Omega = \Omega_c$ the correlation length diverges and $\langle s_j^x s_{j+n}^x \rangle$ becomes scale invariant, i.e., it goes to zero for large n according to a power-law ($\langle s_j^x s_{j+n}^x \rangle \sim n^{-\frac{1}{4}}$). (vi) As can be seen from the above results the set of the critical exponents for the quantum phase transition in dimension one is identical to that for the thermal phase transition of the square-lattice Ising model.

Recently several authors (see, e.g., Refs. [7, 8, 9, 10, 11, 12, 13, 14]) have addressed the question, how deviations from the pure uniform crystalline system (e.g., disorder, aperiodically or regularly varying parameters) influence the properties of quantum spin systems. In what follows we study this question for the transverse Ising model with fields and interactions which vary regularly from site to site with a finite period p, i.e., we change in (1) $\Omega \to \Omega_n$, $I \to I_n$ and the sequence of parameters along the chain is $\Omega_1 I_1 \Omega_2 I_2 \ldots \Omega_p I_p \Omega_1 I_1 \Omega_2 I_2 \ldots \Omega_p I_p \ldots$. In particular, we examine the influence of the so-defined regular inhomogeneity on the existence and the nature of the quantum phase transition. In order to study the properties of the regularly alternating transverse Ising chain we use i) exact analytical results for thermodynamic quantities obtained by means of continued fractions [15], ii) exact analytical results for the GS wave function for special parameter values, iii) exact numerical data for long chains of up to $N = 900$ spins [16], and iv) exact diagonalization data for short chains of $N = 20$ spins.

As described in [15] the thermodynamic quantities of the regularly alternating transverse Ising chain can be calculated rigorously for any finite period of inhomogeneity p. After using the Jordan-Wigner fermionization the transverse Ising chain is described by noninteracting spinless fermions with the Hamiltonian $H = \sum_k \Lambda_k \left(\eta_k^+ \eta_k - \frac{1}{2} \right)$ and the distribution of the square of energy of the elementary excitations $R(E^2) = \frac{1}{N} \sum_k \delta \left(E^2 - \Lambda_k^2 \right)$ can be found exactly by means of continued fractions. The resulting density of states has the form

$$R(E^2) = \begin{cases} \frac{1}{p\pi} \frac{|\mathcal{Z}_{p-1}(E^2)|}{\sqrt{\mathcal{A}_{2p}(E^2)}}, & \text{if} \quad \mathcal{A}_{2p}(E^2) > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where $\mathcal{Z}_{p-1}(E^2)$ and $\mathcal{A}_{2p}(E^2) = -\prod_{j=1}^{2p} \left(E^2 - a_j \right)$ are polynomials of $(p-1)$th and $(2p)$th orders, respectively, and the $a_j \geq 0$ are the roots of $\mathcal{A}_{2p}(E^2)$. For example, $\mathcal{Z}_1(E^2) = 2E^2 - \Omega_1^2 - \Omega_2^2 - I_1^2 - I_2^2$, $\mathcal{A}_4(E^2) = 4\Omega_1^2 \Omega_2^2 I_1^2 I_2^2 - \left(E^4 - \left(\Omega_1^2 + \Omega_2^2 + I_1^2 + I_2^2 \right) E^2 + \Omega_1^2 \Omega_2^2 + I_1^2 I_2^2 \right)^2$ (for details of calculation see [15]). $R(E^2)$ permits to obtain the energy gap Δ for the spin model, since the smallest root of $\mathcal{A}_{2p}(E^2)$ yields the squared minimal excitation energy of the spin system. Knowing $R(E^2)$ we immediately get all GS quantities which follow from the GS energy per site $e_0 = - \int_0^\infty \mathrm{d}E E^2 R(E^2)$. Assuming that $\Omega_n = \Omega + \Delta \Omega_n$ we obtain the GS transverse magnetization $m^z = \frac{\partial e_0}{\partial \Omega}$ and the GS static transverse susceptibility $\chi^z = \frac{\partial m^z}{\partial \Omega}$. The thermodynamic quantities at nonzero temperature follow from the Helmholtz free energy per site $f = -2kT \int_0^\infty \mathrm{d}E E R(E^2) \ln \left(2 \cosh \frac{E}{2kT} \right)$.

Within the analytical approach based on continued fractions it is not possible to calculate the spin correlation

2

functions. However, we can calculate the GS spin correlation functions $\langle s_j^\alpha s_{j+n}^\alpha \rangle$ $(\alpha = x, z)$ numerically for finite chains. It is a specific feature of the considered quantum spin model that the $2^N \times 2^N$ problem inherent in the exact diagonalization scheme reduces to a $N \times N$ problem [6, 16, 2] and therefore numerical exact data for rather long chains can be obtained. (Notice, that typical chain length accessible for Lanczos diagonalization of quantum Heisenberg systems with low symmetry is about $N = 24$.) The on-site magnetizations can be calculated from the correlation functions using the relation $m_{j_1}^\alpha m_{j_2}^\alpha = \lim_{r \to \infty} \langle s_{j_1}^\alpha s_{j_2+rp}^\alpha \rangle$, where $1 \ll j_1 \ll N$ is one of the p consecutive numbers sufficiently far from the ends of chain and $j_2 - j_1 = 0, 1, \ldots, p - 1$. Assuming that $\langle s_j^\alpha s_{j+rp}^\alpha \rangle - \langle s_j^\alpha \rangle \langle s_{j+rp}^\alpha \rangle \sim \exp\left(-\frac{rp}{\xi^\alpha}\right)$ for large r $(1 \ll j, j + rp \ll N)$ we can determine the correlation length ξ^α from the exact numerical data.

Let us discuss the effects of regular alternation on the existence, the position and the critical properties of the quantum phase transition in the transverse Ising chain in more detail. For a certain set of parameters we identify the points of quantum phase transition by looking for vanishing excitation gap Δ. As mentioned above Δ is determined by the smallest root of $\mathcal{A}_{2p}(E^2)$ and therefore the location of a quantum phase transition is determined by $\mathcal{A}_{2p}(0) = 0$. For period $p = 2$ the corresponding condition $\mathcal{A}_4(0) = 0$ yields

$$\Omega_1 \Omega_2 = \pm I_1 I_2. \tag{3}$$

It should be mentioned that a condition for closing of the excitation gap for an inhomogeneous transverse Ising chain was already reported in Ref. [17]. Eq. (3) agrees with that result for the regularly alternating chain of period 2. In what follows we consider as an example a chain with $\Omega_{1,2} = \Omega \pm \Delta\Omega$, $\Delta\Omega \geq 0$. We find either two characteristic fields $\Omega_c = \pm\sqrt{\Delta\Omega^2 + |I_1 I_2|}$ if $\Delta\Omega < \sqrt{|I_1 I_2|}$ or four characteristic fields $\Omega_c = \pm\sqrt{\Delta\Omega^2 \pm |I_1 I_2|}$ if $\Delta\Omega > \sqrt{|I_1 I_2|}$ [18]. This is illustrated in Figs. 1a, 1d where the gap Δ in dependence on Ω (dotted curves) for the chains with $|I_1| = |I_2| = 1$ and $\Delta\Omega = 0.5$ (Fig. 1a) and $\Delta\Omega = 1.5$ (Fig. 1d) is shown. Thus, in case of small strength of transverse-field inhomogeneity $\Delta\Omega$ only quantitative deviations from the homogeneous case may be expected. Contrary, strong inhomogeneity $\Delta\Omega$ increases the number of quantum phase transitions [19]. Using the explicit expression for $\mathcal{A}_4(E^2)$ we find that the smallest root a_j decays as $\sim (\Omega - \Omega_c)^2$ for $\Omega \to \Omega_c$. Therefore, the energy gap Δ closes as $\sim |\Omega - \Omega_c|$ for $\Omega \to \Omega_c$, whereas the GS energy e_0 contains the term $\sim (\Omega - \Omega_c)^2 \ln|\Omega - \Omega_c|$ and hence m^z and χ^z contain the terms $\sim (\Omega - \Omega_c) \ln|\Omega - \Omega_c|$ and $\sim \ln|\Omega - \Omega_c|$, respectively. This means that the critical behaviour of the regularly alternating transverse Ising chain is identical to that of the homogeneous chain. In Figs. 1a, 1d we show m^z vs. Ω (solid curves) and χ^z vs. Ω (dashed curves) for $\Delta\Omega = 0.5$ (Fig. 1a) and $\Delta\Omega = 1.5$ (Fig. 1d). We emphasize, that the quantum phase transition is not suppressed by the deviation of the uniformity in kind of a regular alternation of bonds and fields. Even the appearance of additional transitions due to field alternation is possible.

In addition to the exact results for thermodynamic quantities we can obtain exact expressions for the GS $|\text{GS}\rangle$ of the considered chain at special values of Ω. Obviously, we have $|\text{GS}\rangle = \prod_n | \uparrow_n \rangle$ $(|\text{GS}\rangle = \prod_n | \downarrow_n \rangle)$ for $\Omega \to -\infty$ $(\Omega \to +\infty)$, $|\text{GS}\rangle = \ldots | \downarrow_n \rangle | \uparrow_{n+1} \rangle \ldots$ for $\Omega = 0$ and $\Delta\Omega \gg |I_1|, |I_2|$ and $|\text{GS}\rangle = \prod_n \frac{1}{\sqrt{2}} (| \downarrow_n \rangle + | \uparrow_n \rangle)$ or $|\text{GS}\rangle = \prod_n \frac{1}{\sqrt{2}} (| \downarrow_n \rangle - | \uparrow_n \rangle)$ for $\Omega = \Delta\Omega = 0$ (we have assumed the ferromagnetic sign of the exchange interactions). Nontrivial but solvable is the case $\Omega = \mp\Delta\Omega$, i.e., $\Omega_1 = 0$, $\Omega_2 = -2\Delta\Omega$ or $\Omega_1 = 2\Delta\Omega$, $\Omega_2 = 0$.

Supposing $I_1 I_2 > 0$ the GS for $\Omega_n = 0$, $\Omega_{n+1} = -2\Delta\Omega$ is given by

$$|\text{GS}\rangle = U^+ R^+ \ldots (c_1|\downarrow_{n+1}\rangle|\downarrow_{n+2}\rangle + c_2|\uparrow_{n+1}\rangle|\uparrow_{n+2}\rangle)(c_1|\downarrow_{n+3}\rangle|\downarrow_{n+4}\rangle + c_2|\uparrow_{n+3}\rangle|\uparrow_{n+4}\rangle)\ldots, \tag{4}$$

$$U^+ = \ldots \exp\left(-\mathrm{i}\pi s_n^x s_{n+1}^y\right) \exp\left(-\mathrm{i}\pi s_{n-1}^x s_n^y\right)\ldots,$$

$$R^+ = \ldots \exp\left(-\mathrm{i}\frac{\pi}{2}s_n^z\right)\exp\left(-\mathrm{i}\frac{\pi}{2}s_{n-1}^z\right)\ldots,$$

$$c_1 = \frac{1}{\sqrt{2}}\frac{I + \sqrt{I^2 + \Delta\Omega^2}}{\sqrt{I^2 + I\sqrt{I^2 + \Delta\Omega^2} + \Delta\Omega^2}}, \quad c_2 = \frac{1}{\sqrt{2}}\frac{\Delta\Omega}{\sqrt{I^2 + I\sqrt{I^2 + \Delta\Omega^2} + \Delta\Omega^2}}, \quad I = \frac{I_1 + I_2}{2}.$$

To obtain (4) we note that the transformed Hamiltonian $RUHU^+ R^+$ for $N \to \infty$ (when boundary terms become negligible) describes a system of noninteracting two-site clusters each with the Hamiltonian $I_1 s_{n+1}^z + I_2 s_{n+2}^z - 4\Delta\Omega s_{n+1}^x s_{n+2}^x$. For $\Omega_n = 2\Delta\Omega$, $\Omega_{n+1} = 0$ the GS is again given by (4) with the change $n \to n-1$, $\Delta\Omega \to -\Delta\Omega$. Knowing $|\text{GS}\rangle$ we can calculate the xx and zz spin correlation functions. For example, for $\Omega = -\Delta\Omega$ according to Eq. (4) we find

$$\langle s_{n+1}^x s_{n+2}^x\rangle = \langle s_{n+1}^x s_{n+4}^x\rangle = \ldots = \langle s_{n+2}^x s_{n+3}^x\rangle = \langle s_{n+2}^x s_{n+5}^x\rangle = \ldots = \frac{1}{4}\left(c_2^2 - c_1^2\right),$$

$$\langle s_{n+1}^x s_{n+3}^x\rangle = \langle s_{n+1}^x s_{n+5}^x\rangle = \ldots = \frac{1}{4}\left(c_2^2 - c_1^2\right)^2, \quad \langle s_{n+2}^x s_{n+4}^x\rangle = \langle s_{n+2}^x s_{n+6}^x\rangle = \ldots = \frac{1}{4};$$

$$\langle s_{n+1}^z s_{n+2}^z\rangle = \langle s_{n+1}^z s_{n+4}^z\rangle = \ldots = \langle s_{n+2}^z s_{n+3}^z\rangle = \langle s_{n+2}^z s_{n+5}^z\rangle = \ldots = \langle s_{n+2}^z s_{n+4}^z\rangle = \langle s_{n+2}^z s_{n+6}^z\rangle = \ldots = 0,$$

$$\langle s_{n+1}^z s_{n+3}^z\rangle = \langle s_{n+1}^z s_{n+5}^z\rangle = \ldots = \left(c_1 c_2\right)^2. \tag{5}$$

For $\Omega = \Delta\Omega$ the correlation functions are given by (5) with the change $n \to n-1$.

Now we discuss the exact numerical results for finite chains presented in Fig. 1 and Fig. 2. The order parameters (longitudinal sublattice magnetizations) $|m_j^x|$ versus transverse field Ω are shown in Figs. 1 and 2 for $\Delta\Omega < \sqrt{|I_1 I_2|}$ (Figs. 1b, 2a (open symbols)) and $\Delta\Omega > \sqrt{|I_1 I_2|}$ (Figs. 1e, 2a (full symbols)). In accordance with the analytical results for Δ, m^z and χ^z the numerical data show the existence of either two phases (quantum Ising phase for $|\Omega| < \sqrt{\Delta\Omega^2 + |I_1 I_2|}$ and strong-field quantum paramagnetic phase otherwise) or three phases (low-field quantum paramagnetic phase for $|\Omega| < \sqrt{\Delta\Omega^2 - |I_1 I_2|}$, quantum Ising phase for $\sqrt{\Delta\Omega^2 - |I_1 I_2|} < |\Omega| < \sqrt{\Delta\Omega^2 + |I_1 I_2|}$ and strong-field quantum paramagnetic phase otherwise). The longitudinal on-site magnetizations m_j^x are nonzero in quantum Ising phases and become strictly zero in quantum paramagnetic phases. The transverse magnetization m^z in quantum paramagnetic phases is almost constant being in the vicinity of zero in the low-field phase and of saturation value in the strong-field phase thus producing plateau-like steps in the dependence m^z versus Ω. The correlation length ξ^x (Figs. 1c, 1f, 2b (full symbols)) illustrates that the transition between different phases is accompanied by the divergency of ξ^x. As can be seen in Figs. 1b, 1e short chains are sufficient to reproduce the correct order-parameter behaviour in the quantum Ising phase away from the quantum critical point, but not in the quantum paramagnetic phases. The chain length of up to $N = 900$ sites is clearly sufficient to see a sharp transition in the order parameter (Figs. 1b, 1e) and even to extract reliable results for ξ^x from the long-distance behaviour of $\langle s_j^x s_{j+r}^x\rangle$ as can be seen from the data presented in Figs. 1c, 1f and in Fig. 2b. We note that the numerical data for $|m_j^x|$ and ξ^x at $\Omega = \mp\Delta\Omega$ coincide with the exact expression (5). In particular, the values of the longitudinal sublattice magnetizations for these fields are $\frac{1}{2}$ and $\frac{1}{2}|c_2^2 - c_1^2|$. It should be remarked that inhomogeneity inherent in the model is regular but not random and there is no frustration; as a result we do not observe any indication of a glassy behaviour.

Finally, the low-temperature dependence of the specific heat c at different Ω (≥ 0) obtained from the exact analytical expression for the free energy (see above) confirms the existence of either two phase transitions (Fig. 3a) or four phase transitions (Fig. 3b) depending on the relationship between $\Delta\Omega$ and $\sqrt{|I_1 I_2|}$. At the quantum phase transition points the spin chain becomes gapless and c depends linearly on T. The ridges seen in Figs. 3a, 3b (which correspond to maxima in the dependence c vs. Ω as T varies) single out the boundaries of quantum critical regions [2]. These boundaries correspond to a relation $\Delta \sim kT$ that can be checked by comparison with the data for Δ vs. Ω reported in Figs. 1a, 1d. As can be seen from Fig. 3 the $c(T)$ behaviour for Ω slightly above or below Ω_c changes crossing the boundaries of quantum critical regions. Furthermore we notice that for $\Delta\Omega = 1.5$ an additional low-temperature peak appears in $c(T)$.

To end up, let us turn to a chain of period 3 for which the analytical and numerical calculations presented above can be repeated. The quantum phase transition points follow from the condition $\mathcal{A}_6(0) = 0$, i.e., $\Omega_1 \Omega_2 \Omega_3 = \pm I_1 I_2 I_3$. Depending on the parameters either two, four, or six quantum phase transitions are possible. This behaviour is illustrated in Fig. 4 for a chain of period 3 with $|I_1| = |I_2| = |I_3| = 1$, $\Omega_n = \Omega + \Delta\Omega_n$, $\Delta\Omega_1 + \Delta\Omega_2 + \Delta\Omega_3 = 0$.

To summarize, we present exact results for the thermodynamics of a regularly alternating $s = \frac{1}{2}$ Ising chain in a transverse field. Furthermore, for special parameter values also the exact GS and spin correlation functions are given. For parameters for which the correlation functions are not accessible for rigorous analysis we present exact numerical results for finite but large systems. From these numerical data we can extract the order parameter and the correlation length in high precision. We find, that the quantum phase transition present in the uniform chain is not suppressed by deviation from uniformity in kind of a regular alternation of bonds and fields. On the contrary, field alternation may lead to the appearance of additional transitions. The number of phase transitions for a given period of alternation strongly depends on the precise values of the parameters of the model. For the gap Δ, the GS energy e_0, the transverse magnetization m_z and the static transverse susceptibility χ_z we can examine the critical behaviour exactly and find the same critical indices for the nonuniform and the uniform chain. The presented study can be extended for examining the ground state properties of $s = \frac{1}{2}$ transverse Ising model on one-dimensional superlattices.

References

[1] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Rev. Mod. Phys. **69**, 315 (1997).

[2] S. Sachdev, Quantum Phase Transitions (Cambridge University Press: New York, 1999).

[3] P. Sindzingre, C. Lhuillier, and J. - B. Fouet, cond-mat/0110283.

[4] J. Richter, S. E. Krüger, D. J. J. Farnell, and R. F. Bishop, in Series on Advances in Quantum Many-Body Theory, Vol. 5 (World Scientific, 2001), p.239 (see also cond-mat/0105517).

[5] P. Pfeuty, Ann. Phys. **57,** 79 (1970).

[6] B. K. Chakrabarti, A. Dutta, and P. Sen, Quantum Ising Phases and Transitions in Transverse Ising Models (Springer: Berlin, 1996).

[7] D. S. Fisher, Phys. Rev. B **51**, 6411 (1995).

[8] O. Derzhko and J. Richter, Phys. Rev. B **55**, 14298 (1997).

[9] F. Iglói, L. Turban, D. Karevski, and F. Szalma, Phys. Rev. B **56,** 11031 (1997).

[10] C. Pich, A. P. Young, H. Rieger, and N. Kawashima, Phys. Rev. Lett. **81**, 5916 (1998).

[11] O. Derzhko, J. Richter, and O. Zaburannyi, Physica A **282**, 495 (2000).

[12] F. F. B. Filho, J. P. de Lima, and L. L. Gonçalves, J. Magn. Magn. Mater. **226-230,** 638 (2001).

[13] P. Tong and M. Zhong, Physica B **304,** 91 (2001).

[14] F. Ye, G. - H. Ding, and B. - W. Xu, cond-mat/0105584.

[15] O. Derzhko, J. Phys. A **33,** 8627 (2000).

[16] O. Derzhko and T. Krokhmalskii, phys. stat. sol. (b) **208,** 221 (1998).

[17] P. Pfeuty, Phys. Lett. A **72,** 245 (1979).

[18] Note, that the critical condition found in Ref. [17] (Eq. (6) of that paper) in our notations reads $(\Omega + \Delta\Omega)(\Omega - \Delta\Omega) = I_1 I_2$ and can yield only two critical fields. However, by symmetry arguments (i.e., performing simple rotations of spin axes) one concludes that in addition to that condition there is an excitation of zero energy when $(\Omega + \Delta\Omega)(\Omega - \Delta\Omega) = -I_1 I_2$ and thus either two or four critical fields may appear.

[19] We may consider also a chain of period 2 with $I_{1,2} = I \pm \Delta I$, $\Delta I \geq 0$ and examine the GS properties as I varies. Then we find that for small $\Delta I < \sqrt{|\Omega_1 \Omega_2|}$ there are two values of $I_c = \pm\sqrt{\Delta I^2 + |\Omega_1 \Omega_2|}$, at which the system becomes gapless, whereas for large $\Delta I > \sqrt{|\Omega_1 \Omega_2|}$ there are four such values of $I_c = \pm\sqrt{\Delta I^2 \pm |\Omega_1 \Omega_2|}$. In general, the condition (3) may be tuned by some parameter(s) influencing the local fields or/and exchange interactions.

FIGURE CAPTIONS

Figure 1. The GS properties of the transverse Ising chain of period 2 with $|I_1| = |I_2| = 1$ and $\Omega_{1,2} = \Omega \pm \Delta\Omega$ in dependence on the transverse field Ω: Energy gap Δ (dotted curves) (a,d), transverse magnetization m^z (solid curves) (a,d), static transverse susceptibility χ^z (dashed curves) (a,d), longitudinal sublattice magnetizations $|m_j^x|$ (triangles and solid curves correspond to the data for $N = 20$ and $N = 600$, respectively) (b,e), and correlation

length ξ^x (triangles and squares correspond to the data for $N = 300$ and $N = 600$, respectively) (c,f) for $\Delta\Omega = 0.5$ (a,b,c) and $\Delta\Omega = 1.5$ (d,e,f).

Figure 2. The GS properties of the transverse Ising chain of period 2 with $|I_1| = |I_2| = 1$ and $\Omega_{1,2} = \Omega \pm \Delta\Omega$ in dependence on the transverse field Ω: Longitudinal sublattice magnetizations $|m_j^x|$ (a) and correlation length ξ^x (b) for chain length of $N = 300$ (triangles), $N = 600$ (squares), $N = 900$ (circles) sites and $\Delta\Omega = 0.99$ (open symbols), $\Delta\Omega = 1.01$ (full symbols).

Figure 3. The low-temperature dependence of the specific heat for the transverse Ising chain of period 2 with $|I_1| = |I_2| = 1$ and $\Omega_{1,2} = \Omega \pm \Delta\Omega$; $\Delta\Omega = 0.5$ (a), $\Delta\Omega = 1.5$ (b).

Figure 4. Number of quantum phase transitions present in the transverse Ising chain of period 3 with $|I_1| = |I_2| = |I_3| = 1$ and $\Omega_n = \Omega + \Delta\Omega_n$, $\Delta\Omega_1 + \Delta\Omega_2 + \Delta\Omega_3 = 0$. The dark, grey, or light regions correspond to parameters where the system exhibits two, four, or six quantum phase transitions, respectively.
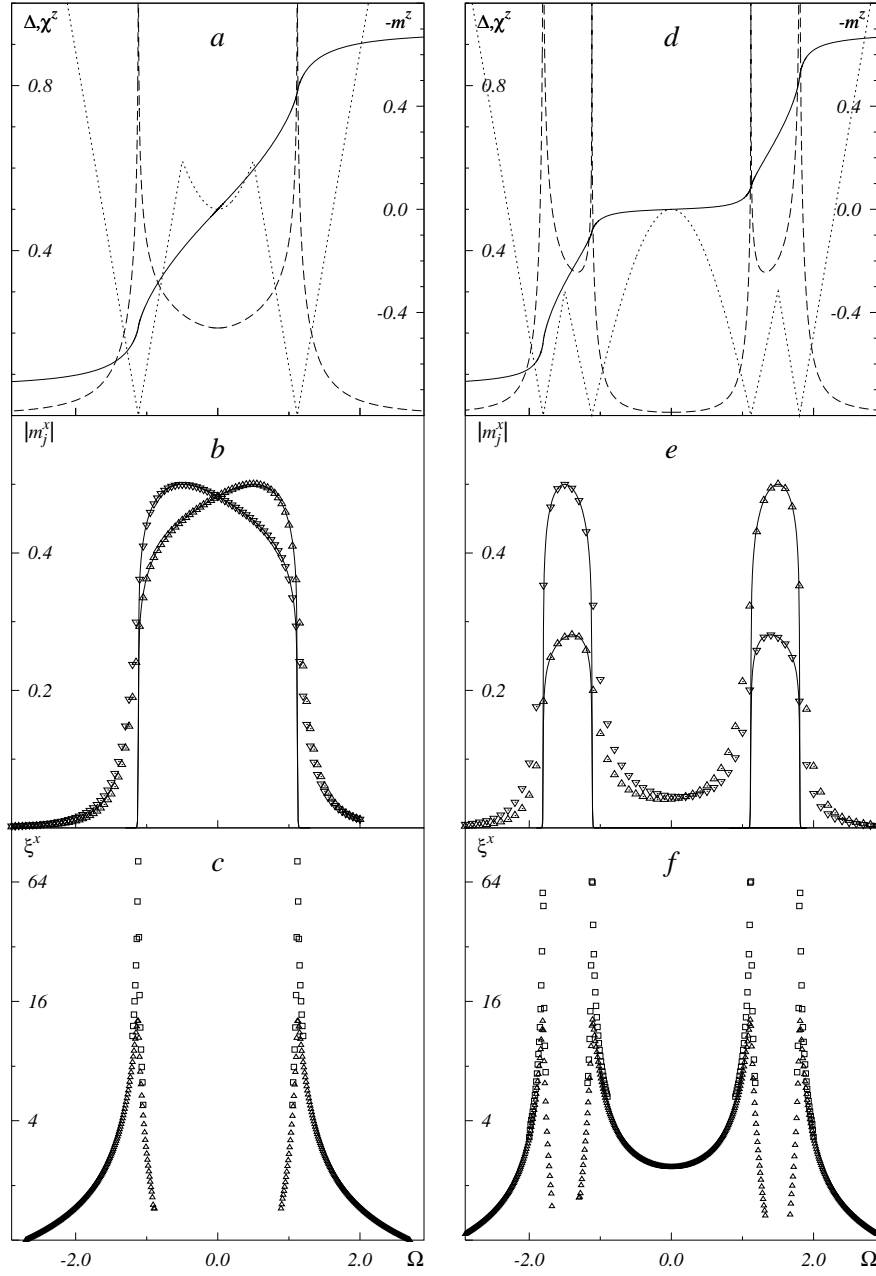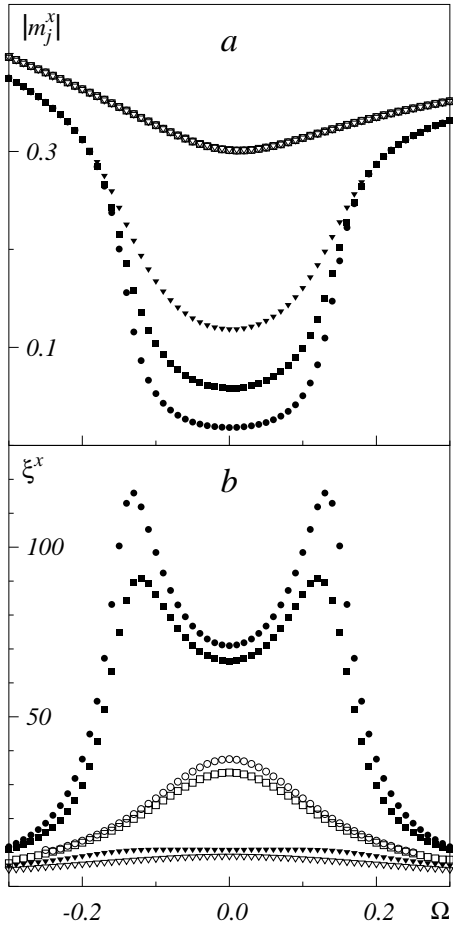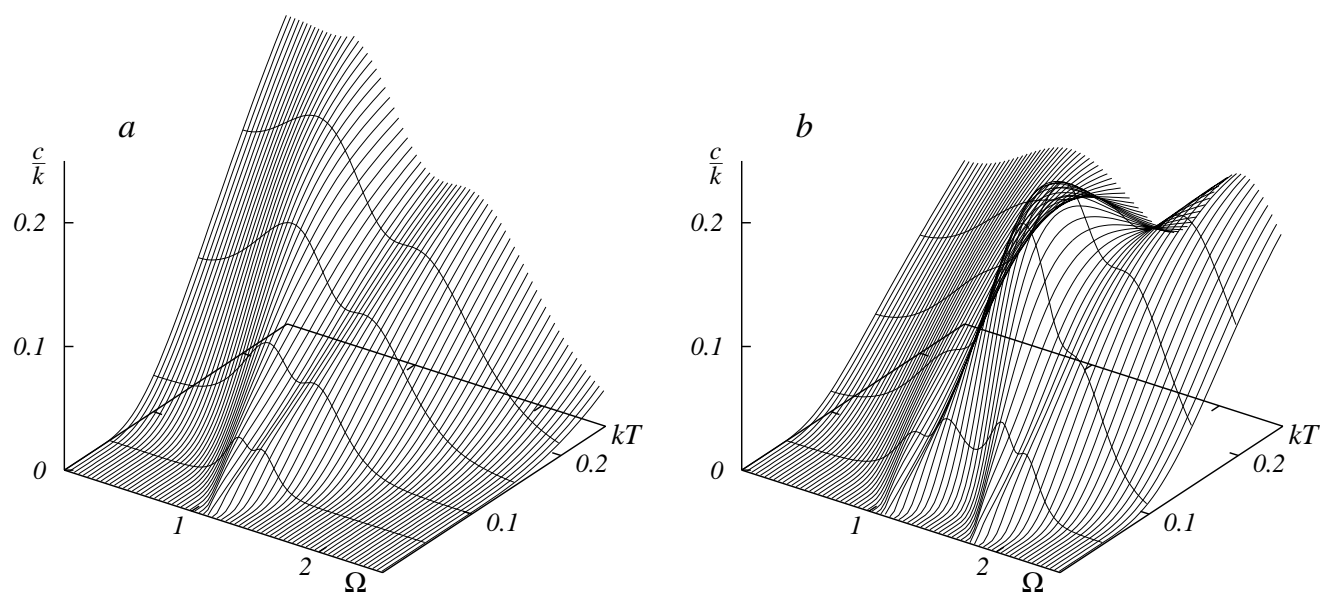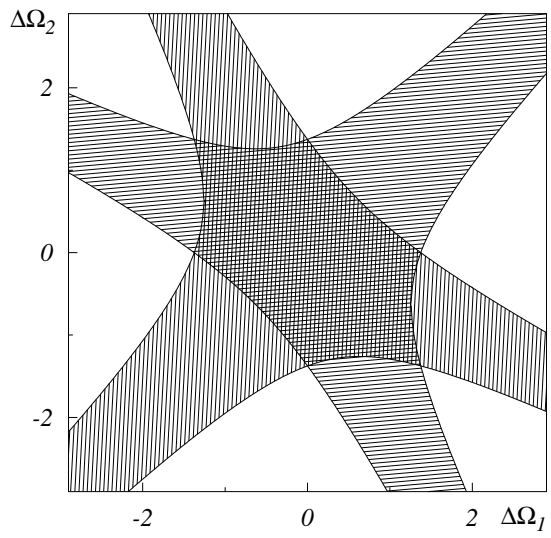
Figure 1:

Figure 2:

Figure 3:

Figure 4: